# Flexural Stress Distribution near a Sharp Crack

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The evaluation of critical crack lengths in structural members has been correlated recently to loading parameters[1] that are derivable from the theory of elasticity. Fundamentally, the cause of rapid crack extension under static loads is attributed to the singular character of the stress distribution in the vicinity of a crack point. As a result, elastic stress analyses of cracked bodies for various loadings and configurations have received increasing attention, particularly in the analysis of fracture.[2, 3]

Now, in order to apply the forementioned fracture concepts to the flexure of cylinders containing crack-like imperfections, it is pertinent to acquire a knowledge of their crack-tip stress field. The main purpose of this note, therefore, is to provide that information.

The problem will be formulated in terms of the classical Saint-Venant flexure function,[4] $\Phi(x,y)$, where the associated twist is assumed to vanish for symmetric cross sections in the interest of clarity. In this case, the field equations are given by

$$\tau_{zx} = -\frac{W}{2(1+\nu)I}\left[\frac{\partial\Phi}{\partial x} + \frac{1}{2}\nu x^2 + \left(1 - \frac{1}{2}\nu\right)y^2\right]$$

$$\tau_{zy} = -\frac{W}{2(1+\nu)I}\left[\frac{\partial\Phi}{\partial y} + (2+\nu)xy\right] \qquad (1)$$

Here, $\nu$ is the Poisson's ratio, $W$ the terminal load directed parallel to one of the principal axes through the centroid; and $I$ the moment of inertia about an axis perpendicular to the load. Although Eq. (1) is only a special case of the general flexure problem,[5] the characteristics of the stress singularities may be found with no loss in generality.†

The character of the singular stresses may be obtained most conveniently by restricting attention to a small region surrounding the crack point and expressing the stress components in polar coordinates (Fig. 1). Hence, Eq. (1) becomes

$$\tau_{zx} = -\frac{W}{2(1+\nu)I}\left(\frac{\partial\Phi}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\sin\theta\right) + \dots$$

$$(2)$$

$$\tau_{zy} = -\frac{W}{2(1+\nu)I}\left(\frac{\partial\Phi}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\cos\theta\right) + \dots$$

where the nonsingular terms are neglected, since they are inconsequential quantities in fracture analyses. The appropriate form of $\Phi$ now will be determined by the eigenfunction expansion method, which first was used by Williams for solving biharmonic equations in the stretching[6] and bending[7] of cracked plates. A product solution of the form

$$\Phi = \sum_{n=0}^{n=\infty} r^{\lambda_n} F(\theta; \lambda_n) \qquad (3)$$

will satisfy the harmonic equation, $\nabla^2\Phi = 0$, if

$$F(\theta; \lambda_n) = a_n \sin\lambda_n\theta + b_n \cos\lambda_n\theta \qquad (4)$$

Introducing the free crack surface conditions, $\tau_{zy} = 0$ for $\theta = \pm\pi$, the values of the eigenparameter $\lambda_n$ are chosen as the positive roots of the characteristic equation

$$\cot\lambda_n\pi = 0 \qquad (5)$$

Therefore

$$\lambda_n = (2n+1)/2 \qquad n = 0,1,2,\dots \qquad (6)$$

The negative values of $n$ have been excluded from the solution in order that the axial displacement $w$, being linearly dependent on $\Phi$, is finite as $r$ approaches the origin. Upon substitution of Eqs. (3) and (4) into (2), the significant stress components corresponding to the minimum eigenvalue, $\lambda_{\min} = \frac{1}{2}$, are

$$\tau_{zx} = \frac{a_0 W}{4(1+\nu)I(r)^{1/2}}\sin\frac{\theta}{2}$$

$$\tau_{zy} = -\frac{a_0 W}{4(1+\nu)I(r)^{1/2}}\cos\frac{\theta}{2} \qquad (7)$$

Equation (7) implies that the distribution of flexural stress always has the same functional form near the singularity caused by the crack point and that it differs only by a parameter $a_0$, depending on the crack geometry. This result suggests the possibilities for expansion of the current fracture theory to the flexure of cracked cylinders. For example, according to the Griffith-Irwin theory, unstable crack geometry under a given load will correspond to some critical value of $a_0$.

A method for determining the constant $a_0$ is given in an earlier investigation.[8] Numerical examples for computing $a_0$ are left to the reader, who may refer to Shepherd's[9] work for a circular section with two equal radial cracks extending inward from the boundary of the section.

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† Qualitatively, the local stress field will remain unchanged upon application of the general flexure theory, since the singular character of the stresses depends only on the fact that the flexure functions are harmonic. Thus, the results will be affected only in a quantitative manner confined to the loading parameters.

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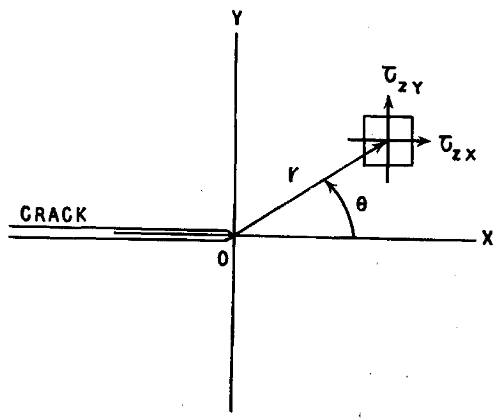


**Fig. 1 Notation for rectangular components of shear near a crack tip**

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# Equilibrium Orientations of Gravity-Gradient Satellites

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The gravitational torque that acts on an orbiting satellite represents an automatic mechanism for attitude control in space. Since configurations sufficiently elongated in the earth-vertical direction experience a positive restoring couple due to the gradient of gravitational attraction when displaced from equilibrium orientation, systems based on this principle can be of a passive character. Practical interest in gravitationally oriented satellites is conditioned by the additional amount of mechanical control which will be required to assure adequate precision of orientation. A calculation is indicated which demonstrates that there is an infinity of equilibrium satellite orientations, in which no control is therefore required. All are in the neighborhood of those discrete orientations for which the gravitational torque is zero.

An equilibrium orientation in space is defined by the directions of orbital angular velocity and instantaneous earth-vertical. If satellite principal inertia axes $x_1$ and $x_3$ occupy these directions, respectively, and $x_2$ is directed opposite to instantaneous linear orbital velocity, the axes form a right-handed triad, gravitational torque is zero, and orbital centrifugal forces are balanced by gravitational attraction. When small angular displacements $\alpha$, $\beta$, $\gamma$ are produced by rotating about $x_1$, $x_2$, $x_3$, respectively, the equilibrium is disturbed and the ensuing angular vibrations about satellite mass center are governed by an extended form of Euler's equations of rigid-body motion:

$$A\ddot{\alpha} + 3\Omega^2(B - C)\alpha = 0 \qquad (1)$$

$$B\ddot{\beta} + 4\Omega^2(A - C)\beta + \Omega(A - B - C)\dot{\gamma} = 0 \qquad (2)$$

$$C\ddot{\gamma} + \Omega^2(A - B)\gamma - \Omega(A - B - C)\dot{\beta} = 0 \qquad (3)$$

Principal-axis moments of inertia are denoted by $A$, $B$, $C$ and $\Omega$ represents orbital angular speed for an orbit assumed for simplicity to be circular; dots indicate differentiations with respect to time. Eqs. (1–3) differ from the forms given in treatises on dynamics, owing to two causes.[1] One is the inclusion on the left-hand sides of the equations of the components of gravitational torque, and the other is due to the complete form of the total moment of momentum for a body in motion about a point *not* fixed in space.

In order to obtain a useful integral of the system of Eqs. (1–3), it is convenient to rewrite them as

$$\ddot{\alpha} = \partial U_1/\partial\alpha \qquad (1')$$

$$\ddot{\beta} + 2a\dot{\gamma} = \partial U_2/\partial\beta \qquad (2')$$

$$\ddot{\gamma} - 2b\dot{\beta} = \partial U_3/\partial\gamma \qquad (3')$$

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where clearly $a$ and $b$ are constants, and $U_1$, $U_2$, $U_3$ are functions only of $\alpha$, $\beta$, $\gamma$, respectively, given by

$$a = \frac{\Omega(A - B - C)}{2B} \qquad b = \frac{\Omega(A - B - C)}{2C}$$

$$U_1(\alpha) = \frac{-3}{2}\Omega^2\frac{(B - C)}{A}\alpha^2 \qquad U_2(\beta) = -2\Omega^2\frac{(A - C)}{B}\beta^2$$

$$U_3(\gamma) = -\frac{1}{2}\Omega^2\frac{(A - B)}{C}\gamma^2$$

Multiplying Eqs. (1′, 2′, and 3′) in turn by $\Omega\dot{\alpha}$, $b\dot{\beta}$, and $a\dot{\gamma}$ and adding, one obtains by means of a single integration

$$(\Omega\dot{\alpha}^2 + b\dot{\beta}^2 + a\dot{\gamma}^2)/2 = \Omega U_1 + bU_2 + aU_3 + K' \qquad (4)$$

where $K'$ is a constant of integration. An equilibrium orientation is recognized by the simultaneous and persistent conditions $\dot{\alpha} = \dot{\beta} = \dot{\gamma} = 0$. Thus if each of the angular velocities $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$ is put equal to zero in Eq. (4), the right-hand side furnishes the geometrical condition that is necessary to secure the corresponding equilibrium orientation. The method employed to obtain the result hardly differs from that used by Jacobi in his treatment of the problem of Lagrange's three particles.[2]

Writing now

$$U(\alpha,\beta,\gamma) = \Omega U_1 + bU_2 + aU_3$$

Eq. (4) reduces to

$$U(\alpha,\beta,\gamma) = -K' \qquad (5)$$

Although the particular orientation given by $\alpha = \beta = \gamma = 0$ is seen from Eqs. (1–3) to represent one orientation of dynamical equilibrium (that one which corresponds to zero gravitational torque), Eq. (5) shows that there is an entire one-parameter family of points in $\alpha,\beta,\gamma$ space for which the same condition is true. For all of the orientations, in general, a net gravitational torque is present but is exactly balanced by the couple generated by the static gradient of centrifugal forces in the orbital motion.

A case of special interest from the practical viewpoint is that in which the satellite possesses axial symmetry about the earth-vertical direction, so that $A = B$. In this case it is easy to see that Eq. (5) reduces to

$$\alpha^2 = \tfrac{2}{3}\beta^2 + K$$

and evidently the constant $K$ must vanish, since $\alpha = 0$, $\beta = 0$ are values corresponding to one solution.

In addition to demonstrating that many equilibrium orientations are to be found in the same neighborhood of attitudes, the present result may be useful in determining advantageous contours for communications satellite reflector disks.

Finally it is noted that the present calculation bridges the gap between two well-known results. Although a spherically symmetric mass distribution possesses a double infinity of equilibrium orientations (i.e., it has no preferred directions), and two point masses attached to a weightless rod possess only six discrete equilibrium orientations (two of which are stable[3]), the calculation shows that for a general distribution of mass there are a single infinity of orientations replacing each of the six discrete ones found by Synge.

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